# Ferrite-Loaded Waveguides – A Perturbation Theory Approach

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*Abstract* — **We develop a perturbation theory for ferrite-loaded waveguides. The perturbation operator is chosen as the deviation of the relative permeability matrix from unity. The method allows a systematic calculation of the propagation constant as well as the perturbed modes to any desired order. It can be used for any direction of the bias field and also for inhomogeneous bias fields as long as the translation invariance along the waveguide is kept. We apply our method to ferrite-loaded waveguides with bias field in propagation direction. For the completely filled case first order expressions for the propagation constant are obtained and the coupling behaviour of TE, TM and TEM modes is discussed. Furthermore we study the scattering of TEM modes in partially filled waveguides and obtain the S-matrix for the ferrite coupled line in first order.**

## I. Introduction

Nonreciprocal devices using ferrite materials are becoming more and more important. E.g. the linearity of transceivers can be substantially improved by using isolators to suppress unwanted reflections. Modern access modes like W-CDMA or EDGE pose very high linearity requirements on the used tranceivers, even for the mobile handsets.

The nonreciprocal effect in ferrite devices results from the interaction between the microwave and the magnetized ferrite. It can be described by introducing a permeability matrix as long as the power level does not become too high [1]. Since first nonreciprocal microwave components were introduced already in the 1950's, there is an extensive literature on the subject [2], [3]. Generally one can distinguish between devices which operate at a frequency far away from the ferromagnetic resonance and devices which work in resonance. The latter usually make use of the high losses which occur close to the resonance. The resonance isolator provides a common example. For devices which work far away from resonance like e.g. a Y-junction circulator, the relative permeability matrix is close to unity. It is therefore reasonable to treat the deviation of the relative permeability tensor from unity as a perturbation and apply perturbation theory to obtain an approximation for the modes of the ferrite loaded structure.

In this work, we apply perturbation theory to the modes of an infinitely long waveguide with fixed cross section. A similar approach is already discussed in [2]. However, here only a first order approximation of the propagation constant is given. The change of the modes itself due to the perturbation is not considered. A coupled mode theory approach has been given by Marcuse [4] and extended to ferrite filled waveguides by Awai and Itoh [5]. Here the special case of two coupled waveguides is considered, i.e. only two modes are taken into account.

In the following we develop a perturbation approach for ferrite loaded waveguides which leads to a systematic expansion of the ferrite-loaded waveguide modes in terms of the unloaded modes. In can be applied if the relative permeability matrix is close to unity. A spatial variation of the permeability matrix within the waveguide cross section is allowed as long as translation invariance along the waveguide is kept. We apply this method to the case of ferrite-loaded waveguides with longitudinal bias field. This configuration might be well suited for monolithic integration of nonreciprocal devices since there are less demagnetisation effects as compared to planar structures with perpendicular bias field (e.g. Y-junction circulator). In case of completely filled waveguides, we obtain the first order change in the propagation constant and give a classification of the different coupling behaviour of TE, TM and TEM modes. Two numerical examples are discussed to compare our method with exact results. Furthermore we study the scattering of TEM modes in partially filled waveguides and obtain the first order expression for the S-matrix. This method is applied to the ferrite coupled line [6] and compared with simulation results.

## II. Perturbation Theory

In the following we assume a time dependence $e^{j\omega t}$ for all fields. The wave equation for the electric field within a ferrite-loaded waveguide reads

$$\mathrm{rot}\hat{\mu}_r^{-1}\mathrm{rot}\vec{E} = k_0^2\vec{E}, \quad (1)$$

where $k_0^2 = \omega^2\epsilon_0\epsilon_r\mu_0$ and $\mathrm{div}\vec{E} = 0$. If the ferrite is biased in z-direction, the permeability tensor is given by

$$\hat{\mu}_r = 1 + \frac{\omega_m}{\omega_0^2 - \omega^2}\begin{pmatrix} \omega_0 & j\omega & 0 \\ -j\omega & \omega_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (2)$$

where $\omega_m = \gamma\mu_0 M$ and $\omega_0 = \gamma\mu_0 H_0$. Here $M$ denotes the magnetisation and $H_0$ the internal magnetic bias field within the ferrite. Clearly $\hat{\mu}_r$ becomes close to 1, if the frequency is far from resonance, i.e. $\omega \ll \omega_0$ or $\omega \gg \omega_0 + \omega_m$. We can then treat the effect of the ferrite loading as a perturbation of the unloaded waveguide. This holds for arbitrary direction of the bias field as well as for inhomogeneous bias fields. The modes of the ferrite-loaded waveguide can then be expanded using the modes of the unloaded waveguide. Far from resonance, the change of the modes up to first order in the perturbation



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is already a good approximation. Many nonreciprocal devices using ferrites operate far from resonance in order to keep the losses small. In those situations, the perturbation method to be described in the following can be savely applied.

In a first step we split the wave equation operator into two parts

$$\mathrm{rot}\hat{\mu}_r^{-1}\mathrm{rot} = \hat{H}_0 + \xi \hat{H}_1, \tag{3}$$

where

$$\hat{H}_0 = \mathrm{rotrot} \tag{4}$$

denotes the wave equation operator for the unperturbed modes and

$$\xi \hat{H}_1 = \mathrm{rot}(\hat{\mu}_r^{-1} - 1)\mathrm{rot} \tag{5}$$

denotes the perturbation operator. We assume that the eigenvectors and eigenvalues of $\hat{H}_0$ are known. The propagation constants of the unperturbed modes result from the requirement, that the eigenvalues are equal to $k_0^2$. The eigenvectors and eigenvalues of the full operator $\hat{H}_0 + \xi\hat{H}_1$ can then be expanded as a series in the small parameter $\xi$. An approximation of the propagation constant to any desired order is given by setting the obtained approximation of the eigenvalue up to this order equal to $k_0^2$. This type of perturbation method is widely used in other physical contexts (see e.g. [7]).

In the following we assume that the waveguide has metal walls and extends in $z$-direction. The cross section of the waveguide as well as the permeability tensor are assumed to be independent of $z$. The eigenvectors of $\hat{H}_0$ can then be labeled as $\vec{E}_{k\beta}^{(0)}$, where $k$ is a discrete index due to the boundary condition for the electric field and $\beta$ refers to the propagation constant. Due to the translation invariance in $z$-direction, $\beta$ is continuous. The eigenvalue equations then read

$$\hat{H}_0 \vec{E}_{k\beta}^{(0)} = (\lambda_k^2 + \beta^2)\vec{E}_{k\beta}^{(0)}. \tag{6}$$

Similarly we denote the eigenvectors of $\hat{H}_0 + \xi\hat{H}_1$ by $\vec{E}_{k\beta}$ and write

$$(\hat{H}_0 + \xi\hat{H}_1)\vec{E}_{k\beta} = (\lambda_k^2 + \beta^2 + \delta_{k\beta})\vec{E}_{k\beta}. \tag{7}$$

The $\vec{E}_{k\beta}^{(0)}$ form a complete orthogonal set and can be classified as TE, TM and TEM modes [8], [9]. TEM modes correspond to $\lambda_k = 0$. We normalize the basis of eigenvectors as

$$\int_V d^3r\, \vec{E}_{k\beta_1}^{(0)\dagger} \vec{E}_{l\beta_2}^{(0)} = 2\pi\delta(\beta_1 - \beta_2)\delta_{kl}, \tag{8}$$

where $V$ denotes the volume of the waveguide. Inserting the expansions

$$\vec{E}_{k\beta} = \vec{E}_{k\beta}^{(0)} + \xi\vec{E}_{k\beta}^{(1)} + \xi^2\vec{E}_{k\beta}^{(2)} + \dots \tag{9}$$

$$\delta_{k\beta} = \xi\delta_{k\beta}^{(1)} + \xi^2\delta_{k\beta}^{(2)} + \dots \tag{10}$$

into (7) and comparing order by order in $\xi$, we find

$$\hat{H}_0 \vec{E}_{k\beta}^{(0)} = (\lambda_k^2 + \beta^2)\vec{E}_{k\beta}^{(0)} \tag{11}$$

$$\hat{H}_0 \vec{E}_{k\beta}^{(1)} + \hat{H}_1 \vec{E}_{k\beta}^{(0)} = (\lambda_k^2 + \beta^2)\vec{E}_{k\beta}^{(1)} + \delta_{k\beta}^{(1)}\vec{E}_{k\beta}^{(0)} \tag{12}$$

$$\vdots$$

We normalize the eigenvectors of $\hat{H}_0 + \xi\hat{H}_1$ as

$$\int_V d^3r\, \vec{E}_{k\beta_1}^{(0)\dagger} \vec{E}_{l\beta_2} = 2\pi\delta(\beta_1 - \beta_2)\delta_{kl}. \tag{13}$$

Then the higher order contributions $\vec{E}_{k\beta}^{(1)}, \vec{E}_{k\beta}^{(2)}, \dots$ do not have a mode contribution of $\vec{E}_{k\beta}^{(0)}$. In the following the solution of the first order equation (12) is discussed.

### *A. First Order Solution – Non-degenerate Case*

We assume that the eigenvector $\vec{E}_{k\beta}^{(0)}$ is not degenerate. In addition we assume the $\hat{H}_1$ does not depend on $z$. In order to obtain a solution of (12), we expand

$$\vec{E}_{k\beta}^{(1)} = \sum_{l\neq k} a_l \vec{E}_{l\beta}^{(0)} \tag{14}$$

and project on the state $\vec{E}_{l\beta}^{(0)}$. This leads to

$$\delta_{k\beta}^{(1)} = \int_F dxdy\, \vec{E}_{k\beta}^{(0)\dagger} \hat{H}_1 \vec{E}_{k\beta}^{(0)} \tag{15}$$

$$a_l = \frac{1}{\lambda_k^2 - \lambda_l^2} \int_F dxdy\, \vec{E}_{l\beta}^{(0)\dagger} \hat{H}_1 \vec{E}_{k\beta}^{(0)}. \tag{16}$$

Here $F$ denotes the cross section of the waveguide. Note that the sum in (14) does not contain a term $k = l$ due to the chosen normalisation in (13). The expansion in (14) can not be applied, if the eigenvector $\vec{E}_{k\beta}^{(0)}$ is degenrate, since the corresponding expansion coefficients in (16) would not be well defined.

### *B. First Order Solution – Degenerate Case*

If $\lambda_k$ is r-fold degenerate, we start the perturbation expansion from appropriate linear combinations of the degenerate eigenvectors $\vec{E}_{k_i\beta}^{(0)}, i = 1, \dots, r$. Let

$$M_{ij} = \int_F dxdy\, \vec{E}_{k_i\beta}^{(0)\dagger} \hat{H}_1 \vec{E}_{k_j\beta}^{(0)} \tag{17}$$

and

$$\hat{M}\vec{v}^{(q)} = \delta^{(q)}\vec{v}^{(q)}. \tag{18}$$

The coefficients of the normalized eigenvectors $\vec{v}^{(q)}$ form the required linear combinations

$$\vec{E}_{k\beta}^{(q)(0)} = \sum_{i=1}^{r} v_i^{(q)} \vec{E}_{n_i\beta}^{(0)}, \quad q = 1, \dots, r. \tag{19}$$

As in the non-degenerate case, we expand

$$\vec{E}_{k\beta}^{(q)(1)} = \sum_{l\neq k} a_l \vec{E}_{l\beta}^{(0)} \tag{20}$$

and find

$$\delta_{k\beta}^{(q)(1)} = \int_F dxdy\, \vec{E}_{k\beta}^{(q)(0)\dagger} \hat{H}_1 \vec{E}_{k\beta}^{(q)(0)} \tag{21}$$

$$a_l = \frac{1}{\lambda_k^2 - \lambda_l^2} \int_F dxdy\, \vec{E}_{l\beta}^{(0)\dagger} \hat{H}_1 \vec{E}_{k\beta}^{(q)(0)}. \tag{22}$$

## III. Completely Filled Waveguides with Longitudinal Bias Field – First Order Results

We apply the perturbation method described in the previous section to the case where the waveguide is completely filled with ferrite material and biased in $z$-direction. The coupling behaviour of the modes is encoded in the coupling coefficients

$$C_{kl}^{(s)(t)} = \int_F dxdy\, \vec{E}_{k\beta}^{(s)\dagger} \xi \hat{H}_1 \vec{E}_{l\beta}^{(t)}, \quad \text{s,t} \in \{\text{te,tm,tem}\}. \quad (23)$$

Taking the special properties of TE, TM and TEM modes [9] into account we derive the following first order results

1. The propagation constants of nondegenerate modes are given as
$$\beta_k^{(\text{te})2} = \mu_{\text{eff}}(k_0^2 - \lambda_k^{(\text{te})2}) \quad (24)$$
$$\beta_k^{(\text{tm})2} = \mu_{\text{eff}} k_0^2 - \lambda_k^{(\text{tm})2} \quad (25)$$
$$\beta_k^{(\text{tem})2} = \mu_{\text{eff}} k_0^2, \quad (26)$$
where
$$\mu_{\text{eff}} = \frac{(\omega_0 + \omega_m)^2 - \omega^2}{\omega_0(\omega_0 + \omega_m) - \omega^2}. \quad (27)$$
2. Degenerate TM and TEM modes remain degenerate and the equations above for the propagation constant remain valid. In contrast to this, the degeneracy of TE modes is lifted.
3. The degeneracy of TM and TEM modes can be lifted, e.g. if the waveguide is not completely filled.
4. Coupling between modes is mediated only through TE modes, i.e. TM and TEM modes acquire only additional TE contributions, whereas TE modes acquire additional TE, TM and TEM contributions.

In the following we give two examples which illustrate our findings.

### A. Coaxial Line

The ground mode of the coaxial line is a TEM mode. In first order perturbation theory, we obtain for the propagation constant $\beta = \sqrt{\mu_{\text{eff}}}\, k_0$. This coincides with the Suhl and Walker approximation [10]. In Fig. 1 $\beta/k_0$ is plotted against frequency below resonance. We find good agreement with the exact result obtained in [11] since the second order is strongly suppressed due to the large cut-off frequencies of the higher modes.

### B. Cylindrical Waveguide

The unloaded cylindrical waveguide with circular cross section has two-fold degenerate TE modes, which can be written as

$$\vec{E}_{ni\pm\beta}^{(0)} = \frac{1}{k_{ni}} \text{rot}(\psi_{ni\pm} \mathrm{e}^{-j\beta z}), \quad (28)$$

where

$$\psi_{ni\pm} = \frac{2}{\sqrt{\pi}} \frac{k_{ni}}{\sqrt{(k_{ni}R)^2 - n^2}} \frac{J_n(k_{ni}\rho)}{J_n(k_{ni}R)} \mathrm{e}^{\pm jn\phi}. \quad (29)$$

Here $R$ denotes the radius of the cross section and $J_n'(k_{ni}R) = 0$. The propagation constants are given by $\beta_{ni\pm}^2 = k_0^2 - k_{ni}^2$.

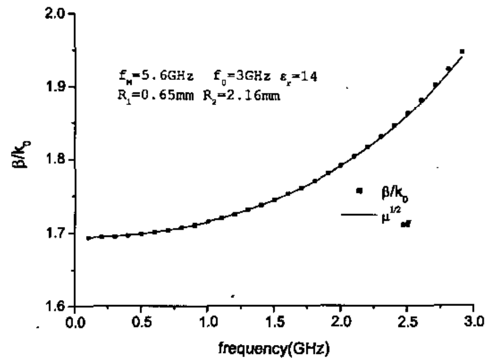


Fig. 1. The Figure shows $\beta/k_0$ plotted against frequency below resonance. The squares correspond to the exact result, the solid line displays our result.

Loading this waveguide with ferrite material with bias field in propagation direction leads to a splitting of those degenerate modes. Applying degenerate perturbation theory leads to the following expression for the propagation constants

$$\beta_{ni\pm}^2 = \frac{\mu_{\text{eff}}(k_0^2 - k_{ni}^2)}{1 \pm \frac{\omega\omega_m}{\omega_0(\omega_0+\omega_m)-\omega^2} \frac{2n}{(k_{ni}R)^2 - n^2}}. \quad (30)$$

This result is plotted in Fig. 2 for frequencies above resonance together with the exact result from [2]. We find excellent agreement.

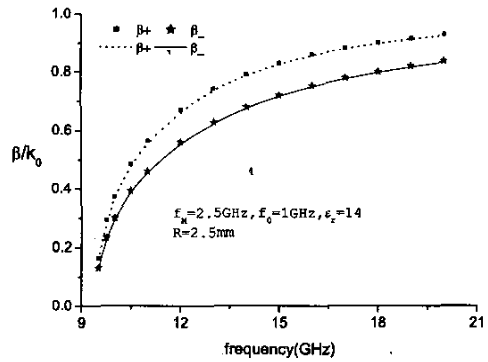


Fig. 2. The diagram shows the propagation constants of the dominant TE modes. The 2-fold degeneracy of the propagation constant is lifted due to the ferrite material. The data points correspond to the exact result whereas the solid and dottet lines show our perturbation result. The frequency range is well above resonance.

## IV. Scattering of TEM Modes – Ferrite Coupled Line

In the following we consider the scattering problem, where the waveguide is only partially filled with ferrite. The ferrite material is assumed to be present only between two interfaces located at $z = z_1$ and $z = z_2$, $z_2 - z_1 = L > 0$. Furthermore we assume that the ferrite has the same permittivity as the surrounding dielectric. If the structure supports $n$ TEM modes

$$\vec{E}_k = \vec{A}_k^{(\mathrm{E})} \mathrm{e}^{-jk_0 z} \tag{31}$$

$$\vec{H}_k = \frac{1}{\eta_0} \vec{A}_k^{(\mathrm{H})} \mathrm{e}^{-jk_0 z}, \tag{32}$$

where $\vec{A}_k^{(\mathrm{E})} = -\vec{\nabla}\psi_k(x,y)$ and $\vec{A}_k^{(\mathrm{H})} = \vec{e}_z \times \vec{A}_k^{(\mathrm{E})}$, the ferrite-loaded modes can be obtained by applying degenerate first order perturbation theory as

$$\vec{E}_k^f = \sum_{l=1}^{n} v_l^{(k)} \vec{A}_l^{(\mathrm{E})} \mathrm{e}^{-j\beta_k z} \tag{33}$$

$$\vec{H}_k^f = \frac{1}{\eta_k} \sum_{l=1}^{n} v_l^{(k)} \vec{A}_l^{(\mathrm{H})} \mathrm{e}^{-j\beta_k z}. \tag{34}$$

Here we have ignored the coupling to any TE and TM modes which is of course only justified, if the working frequency is well below the cut-off frequency of these modes. The first order result of the magnetic field was obtained by the requirement, that the modes remain TEM and that $\mathrm{rot}\vec{H} = j\omega\epsilon_0\epsilon_r\vec{E}$ holds. This is consistent with the neglection of higher TE and TM modes. The propagation constants $\beta_k = k_0/\sqrt{1+\delta_k}$, characteristic impedances $\eta_k = \eta_0/\sqrt{1+\delta_k}$ and expansion coefficients $v_l^{(k)}$ are obtained from the eigenvalues $\delta_k$ and eigenvectors $\vec{v}^{(k)}$ of the coupling matrix

$$M_{kl} = \int_{F_{\mathrm{ferrite}}} dxdy\, \vec{A}_k^{(\mathrm{H})\dagger} (\hat{\mu}_r^{-1} - 1) \vec{A}_l^{(\mathrm{H})}. \tag{35}$$

The reflection/transmission from the unloaded mode $l$ to the unloaded mode $k$ is then given by

$$S_{kl}^{\mathrm{R}} = \sum_{r=1}^{n} \frac{v_k^{(r)*} v_l^{(r)} \frac{j}{2}\left(\frac{\eta_r}{\eta_0} - \frac{\eta_0}{\eta_r}\right)\sin(\beta_r L)}{\cos(\beta_r L) + \frac{j}{2}\left(\frac{\eta_r}{\eta_0} + \frac{\eta_0}{\eta_r}\right)\sin(\beta_r L)} \tag{36}$$

$$S_{kl}^{\mathrm{T}} = \sum_{r=1}^{n} \frac{v_k^{(r)*} v_l^{(r)}}{\cos(\beta_r L) + \frac{j}{2}\left(\frac{\eta_r}{\eta_0} + \frac{\eta_0}{\eta_r}\right)\sin(\beta_r L)}. \tag{37}$$

### A. Ferrite Coupled Line

We apply the method explained above to the half filled ferrite coupled stripline [6] which has two degenerate TEM modes, even (e) and odd (o). The corresponding waveguide cross section is shown in the inset of Fig. 3. The magnitude of the obtained transmission $S_{oo}^{\mathrm{T}} = S_{ee}^{\mathrm{T}}$ and $S_{eo}^{\mathrm{T}} = -S_{oe}^{\mathrm{T}}$ is plotted in Fig. 3 together with simulation data. We find good agreement.

## V. Conclusion

A perturbation theory for ferrite-loaded waveguides was developed. The perturbation operator was chosen as the deviation of the relative permeability matrix from unity. The method allows a systematic calculation of the propagation constant as well as the perturbed modes to any desired order. It can be used for any direction of the bias field and even for inhomogeneous

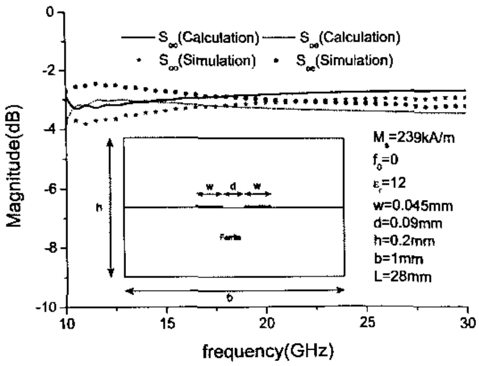


Fig. 3. The diagram shows the transmission odd/odd and even/odd for the half fi lled ferrite coupled line as a function of frequency. The waveguide cross section is shown in the inset. Data points correspond to simulation results whereas the solid and dotted lines display the perturbation result.

bias fields as long as the translation invariance along the waveguide is kept. For the case of the completely filled waveguide it was shown that up to first order in perturbation theory the mixing between modes is only mediated by TE modes. In particular, degeneracy of TM and TEM modes is not lifted. However, if the waveguide is not completely filled, such a mixing in general occurs. Furthermore we derived the S-matrix for the scattering of TEM modes in partially ferrite-loaded waveguides. As an example we discussed the half filled ferrite coupled line and found good agreement between perturbation results and simulation data.

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